Prop: Every bilinear transformation maps circles or straight lines into circles or straight lines. 2013

Proof: Equation of circle with p and q as inverse points in z-plane is given by

$$\left|\frac{z-p}{z-q}\right| = k,\ k \neq 1. \quad \text{In the case of st. line } k=1 \quad \ldots (1)$$

The bilinear transformation

$$w = \frac{az+b}{cz+d} \quad \text{i.e.} \quad z = \frac{dw-b}{-cw+a}$$

transforms (1) into the form

$$\left|\frac{\frac{dw-b}{-cw+a} - p}{\frac{dw-b}{-cw+a} - q}\right| = k.$$

i.e. $$\left|\frac{dw-b-p(-cw+a)}{dw-b-q(-cw+a)}\right| = k$$

i.e. $$\left|\frac{w(d+cp)-(ap+b)}{w(d+cq)-(aq+b)}\right| = k$$

i.e. $$\left|\frac{(d+cp)\left\{w-\frac{ap+b}{d+cp}\right\}}{(d+cq)\left\{w-\frac{aq+b}{d+cq}\right\}}\right| = k$$

i.e. $$\left|\frac{(d+cp)(w-p')}{(d+cq)(w-q')}\right| = k, \quad \text{where } p' = \frac{ap+b}{d+cp},\ q' = \frac{aq+b}{d+cq}$$

$$\therefore p' = \left(\frac{az+b}{d+cz}\right)_{z=p},\ q' = \left(\frac{az+b}{d+cz}\right)_{z=q}$$

i.e. $$\left|\frac{w-p'}{w-q'}\right| = \left|\frac{d+cq}{d+cp}\right| k.$$

i.e. $$\left|\frac{w-p'}{w-q'}\right| = k'. \quad \text{where } k' = \left|\frac{d+cq}{d+cp}\right| \cdot k. \quad \ldots (2)$$

clearly $k' \neq 1$. if

so. Eq^n (2) represents the circle in w-plane with p' and q' as inverse points.

If $k=1$, then (1) represents a st. line which bisect perpendicularly the line joining p to q. The bilinear transformation the line (1) of z plane into a circle w-plane and the points p, q symmetrical about become inverse points in w-plane.

Proved

Problem 1. Find the image of the circle $|z-2|=2$ under the bilinear transformation $w=\frac{z}{z+1}$ 2014-2016, 2012, 2010, 2009

Sol^n: Bilinear transformation is given as

$$w=\frac{z}{z+1} \quad \text{i.e. } wz+w=z.$$

$$\text{i.e. } z(w-1)=-w$$

$$\therefore z=-\frac{w}{w-1}=\frac{w}{1-w}.$$

so, $z=\frac{w}{1-w}$ . . . . (1)

Now circle is given as $|z-2|=2$

$$\text{i.e. } |z-2|^2=4$$

$$\text{i.e. } (z-2)(\bar{z}-2)=4, \text{ as } |z|^2=z.\bar{z}$$

$$\text{i.e. } \left(\frac{w}{1-w}-2\right)\left(\frac{\bar{w}}{1-\bar{w}}-2\right)=4, \text{ Using (1)}$$

$$\text{i.e., } \frac{w-2+2w}{1-w}\cdot\frac{\bar{w}-2+2\bar{w}}{1-\bar{w}}=4.$$

$$\text{i.e., } \cancel{(1-\bar{w})(\bar{w}-2-w)=4-w)(-2-\bar{w})}$$

$$\text{i.e., } (-2-w)(-2-\bar{w})=4(1-w)(1-\bar{w})$$

$$\text{i.e., } (2+w)(2+\bar{w})=4(1-\bar{w}-w+w.\bar{w})$$

$$\text{i.e., } 4+2\bar{w}+2w+w.\bar{w}=4-4\bar{w}-4w+4.w.\bar{w}$$

$$\text{i.e., } (3w-2)(3\bar{w}-2)=4(1-w)(1-\bar{w})=4(1-\bar{w}-w+w\bar{w})$$

$$=4-4\bar{w}-4w+4w.\bar{w}$$

$$\text{i.e., } 9w.\bar{w}-6w-6\bar{w}+4=4-4\bar{w}-4w+4w.\bar{w}$$

$$\text{i.e., } 5w.\bar{w}-2w-2\bar{w}=0.$$

$$\text{i.e., } 5(u^2+v^2)-2(w+\bar{w})=0, \text{ since } w=u+iv$$

$$\therefore \bar{w}=u-iv$$

$$\text{i.e. } 5(u^2+v^2)-4u=0.$$

$$\text{i.e. } u^2+v^2-\frac{4u}{5}=0. \text{ Compare with } x^2+y^2+2gx+2fy+c=0$$

This represents a circle with centre $g=-\frac{2}{5}, f=0, c=0$

$$(-g,-f)=\left(\frac{2}{5},0\right)$$

and radius $=\sqrt{g^2+f^2-c}=\sqrt{\frac{4}{25}}=\frac{2}{5}$ in w-plane.

Hence, image of circle $|z-2|=2$ is a circle with centre $\left(\frac{2}{5},0\right)$ and radius $\frac{2}{5}$ in w plane.

Ans.

Problems on Cross Ratio

Prob. 2. Find the bilinear transformation that maps the points $z_1=\infty, z_2=i, z_3=0$ into the points $w_1=0, w_2=i, w_3=\infty$. resp.

Sol^n: Required transformation is:

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w_4)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\text{i.e. } \frac{(w-0)(i-\infty)}{(0-i)(\infty-w)}=\frac{(z-\infty)(i-0)}{(\infty-i)(0-z)}.$$

$$\text{i.e. } \frac{w}{-i}\cdot\frac{i-\infty}{\infty-w}=\frac{z-\infty}{\infty-i}\cdot\frac{i}{-z}.$$

i.e., $\frac{w}{i}\,\frac{i-\infty}{\infty-w} = \frac{i}{z}\,\frac{z-\infty}{\infty-i}$ . . . (1)

Now $\frac{i-\infty}{\infty-w} = \lim_{n\to\infty} \frac{i-n}{n-w}$ . . ($\frac{\infty}{\infty}$ form)

$= \lim_{n\to\infty} \frac{0-1}{1-0}$, by L-Hospital rule.

$= -1$

And $\frac{z-\infty}{\infty-i} = \lim_{n\to\infty} \frac{z-n}{n-i}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{n\to\infty} \frac{-1}{1} = -1.$

So, transformation (1) reduces to

$\frac{w}{i}(-1) = \frac{i}{z}(-1).$

i.e. $w = \frac{i^2}{z} = -\frac{1}{z}.$

Hence, the required transformation is $w = -\frac{1}{z}$. Ans.

Prob. Find the bilinear transformation which maps the points $z = 0, -1, \infty$ into the points $w = -1, -2, -i, i$ resp. 2011

Solⁿ: Required transformation is:

$$\frac{(w-w_1)}{(w_1-w_2)}\cdot\frac{(w_2-w_3)}{(w_3-w)} = \frac{(z-z_1)}{(z_1-z_2)}\,\frac{(z_2-z_3)}{(z_3-z)}.$$

i.e. $\frac{w-(-1)}{-1-(-2-i)}\cdot\frac{-2-i-i}{i-w} = \frac{z-0}{0-(-1)}\cdot\frac{-1-\infty}{\infty-z}.$

i.e. $\frac{w+1}{1+i}\cdot\frac{-2-2i}{i-w} = \frac{z}{1}\cdot\frac{-1-\infty}{\infty-z}.$

i.e. $\frac{w+1}{1+i}\,\frac{2+2i}{i-w} = z\,\frac{1+\infty}{\infty-z}$

Now $\frac{1+\infty}{\infty+z} = \lim_{n\to\infty}\frac{1+n}{n+z} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{z}{n}} = 1.$

So, $\frac{w+1}{(1+i)}\cdot\frac{2(1+i)}{i-w} = z\cdot1.$

$\therefore\ 2\frac{(w+1)}{i-w} = \cancel{z(1+}\ z.$

i.e. $2w+2 = iz - wz.$

i.e. $w(2+z) = iz-2.$

$\therefore\ w = \frac{iz-2}{z+2}.$

This is the required transformation.

Prob. Find the bilinear transformation which maps the points $z = -2, 0, 2$ into the points $w = 0, i, -i$ resp. 2014

Ans: $w = i\left(\frac{2+z}{2-3z}\right)$

Hint: $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$\frac{(w-0)(i+i)}{(0-i)(-i-w)} = \frac{(z+2)(0-2)}{(-2-0)(2-z)}$

i.e. $\frac{w+i}{w} = \frac{4-2z}{z+2}$ $\therefore\ w = i\left(\frac{2+z}{2-3z}\right).$

Find the linear map for $z = 0, -i, -1$ and corresponding values of w are $w = i, 1, 0$. [2009]

: Hint: $\dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$.

i.e. $\cancel{\dfrac{(w-i)(1-0)}{(i-1)(0-w)}}$

i.e. $\dfrac{(w-i)(1-0)}{(i-1)(0-w)} = \dfrac{(z-0)(-i+1)}{(0+i)(-1-z)}$.

i.e. $\dfrac{w-i}{(i-1)(-w)} = \dfrac{z(1-i)}{i(-1)(1+z)}$.

i.e. $\dfrac{w-i}{(i-1)w} = \dfrac{z(1-i)}{i(1+z)}$

i.e. $\dfrac{w-i}{w} = \dfrac{-z(i-1)(i-1)}{i(z+1)} = \dfrac{-z(i-1)^2}{i(z+1)} = \dfrac{-z(i^2+1-2i)}{i(z+1)}$

$\dfrac{w-i}{w} = \dfrac{-z(-2i)}{i(z+1)} = \dfrac{2z}{z+1}$

$\therefore 1 - \dfrac{i}{w} = \dfrac{2z}{z+1}$

$\therefore 1 - \dfrac{2z}{z+1} = \dfrac{i}{w}$

i.e. $\dfrac{z+1-2z}{z+1} = \dfrac{i}{w}$

$\therefore w = i\,\dfrac{z+1}{1-z}$

So, $w = i\left(\dfrac{1+z}{1-z}\right)$.

———x———

8. Find the fixed point and the normal form of the following bilinear transformations and classify their nature: (i) $w = \dfrac{3z-4}{z-1}$ (ii) $w = \dfrac{z-1}{z+1}$

Sol^n (i) The fixed points are given by:

$$z = \frac{3z-4}{z-1} \quad \text{i.e. } z^2 - z = 3z-4$$
$$\text{i.e. } z^2-4z+4=0$$
$$\Rightarrow (z-2)^2=0$$

So, $z=2$ is the only fixed point.

Now to find normal form, we have:

$$w = \frac{3z-4}{z-1} \qquad wz-2w-2z+4$$

i.e. $wz - w = 3z-4$

i.e. $wz-w-3z+4=0$

i.e. $(w-2)(z-2)+2z+2w-w-3z=0$

i.e. $(w-2)(z-2) - z+w=0$

i.e. $(w-2)(z-2)-(z-2)+(w-2)=0$

i.e. $1-\dfrac{1}{w-2}+\dfrac{1}{z-2}=0$

i.e. $\dfrac{1}{w-2} = \dfrac{1}{z-2}+1$

This is the required normal form.

Here the transformation is parabolic.

(ii) The fixed points are given by:

$$z = \frac{z-1}{z+1} \quad \text{i.e. } z^2+z$$

i.e. $z^2+z = z-1$

$$\Rightarrow z^2+1=0 \Rightarrow z=\pm i$$

Hence $z=i$ and $z=-i$ are the two fixed points.

To obtain normal form, we have:

$$w = \frac{z-1}{z+1}$$

$\therefore$ $w-i = \dfrac{z-1}{z+1}-i$ and $w+i = \dfrac{z-1}{z+1}+i$

$$\therefore \frac{w-i}{w+i} = \frac{z-1-iz-i}{z-1+iz+i} = \frac{z-iz-1-i}{z+iz-1+i} = \frac{z(1-i)-i(1+i)}{z(1+i)+i(1+i)}$$
$$= \frac{(1-i)(z-i)}{(1+i)(z+i)}$$

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$$\therefore \frac{w-i}{w+i} = \frac{(1-i)(1-i)(z-i)}{(1+i)(1-i)(z+i)} = \frac{(1-i)^2(z-i)}{(1-i^2)(z+i)}$$
$$= \frac{(1+i^2-2i)(z-i)}{2(z+i)} = \frac{-2i(z-i)}{2(z+i)}$$
$$= -i \cdot \frac{z-i}{z+i}$$

$$\therefore \frac{w-i}{w+i} = -i \cdot \frac{z-i}{z+i}$$

This is the normal form.

Here: $\lambda = -i$, so $|\lambda|=1$

Hence the transformation is elliptic.